3.3 – Orthogonality

Definition: Two nonzero vectors $\mathbf{u}$ and $\mathbf{v}$ in R^n are said to be **orthogonal** (or **perpendicular**) if $\mathbf{u} \cdot \mathbf{v} = 0$. We will also agree that the zero vector in R^n is orthogonal to every vector in R^n.

Definition: A vector $\mathbf{n}$ that is orthogonal to a line in R^2 or R^3 or a plane in R^3 is called a **normal**.

#4 Find a point-normal form of the equation of the plane passing through P and having $\mathbf{n}$ as a normal.
$P(1, 1, 4)$, $\mathbf{n} = (1, 9, 8)$

Theorem 3.3.1
a) If a and b are constants that are not both zero, then an equation of the form $ax + by + c = 0$ represents a line in R^2 with normal $\mathbf{n} = (a, b)$.
b) If a, b, and c are constants that are not all zero, then an equation of the form $ax + by + cz + d = 0$ represents a plane in R^3 with normal $\mathbf{n} = (a, b, c)$.

Theorem 3.4.3 (not a typo)
If A is an $m \times n$ matrix, then the solution set of the homogeneous linear system $A\mathbf{x} = \mathbf{0}$ consists of all vectors in R^n that are orthogonal to every row vector of A.

$$\begin{array}{ccccccccc} a_{11}x_1 & + & a_{12}x_2 & + & \ldots & + & a_{1n}x_n & = & b_1 \\ a_{21}x_1 & + & a_{22}x_2 & + & \ldots & + & a_{2n}x_n & = & b_2 \\ \vdots & & \vdots & & & & \vdots & & \vdots \\ a_{m1}x_1 & + & a_{m2}x_2 & + & \ldots & + & a_{mn}x_n & = & b_m \end{array}$$

Theorem 3.3.2 Projection Theorem

If $\mathbf{u}$ and $\mathbf{a}$ are vectors in R^n, and if $\mathbf{a} \neq \mathbf{0}$, then $\mathbf{u}$ can be expressed in exactly one way in the form $\mathbf{u} = \mathbf{w}_1 + \mathbf{w}_2$, where $\mathbf{w}_1$ is a scalar multiple of $\mathbf{a}$ and $\mathbf{w}_2$ is orthogonal to $\mathbf{a}$.

Definition: When $\mathbf{u}$, a vector in R^n, is expressed as $\mathbf{u} = \mathbf{w}_1 + \mathbf{w}_2$, where $\mathbf{w}_1$ is a scalar multiple of a vector $\mathbf{a}$ in R^n, $\mathbf{w}_1 = \text{proj}_{\mathbf{a}}\mathbf{u} = \dfrac{\mathbf{u} \cdot \mathbf{a}}{\|\mathbf{a}\|^2}\mathbf{a}$ is called the **orthogonal projection of u on a** or the **vector component of u along a**. The vector $\mathbf{w}_2 = \mathbf{u} - \text{proj}_{\mathbf{a}}\mathbf{u} = \mathbf{u} - \dfrac{\mathbf{u} \cdot \mathbf{a}}{\|\mathbf{a}\|^2}\mathbf{a}$ is called the **vector component of u orthogonal to a**.

#20 Find the vector component of **u** along **a** and the vector component of **u** orthogonal to **a.**
$\mathbf{u} = (5, 0, -3, 7)$, $\mathbf{a} = (2, 1, -1, -1)$

#38 Find the standard matrix for the orthogonal projection of R^2 onto the stated line, and then use that matrix to find the orthogonal projection of the given point onto that line.

The orthogonal projection of $(1, 2)$ onto the line that makes an angle of $\pi/4$ $(= 45^\circ)$ with the positive x-axis.

#36 Find the standard matrix for the reflection of R^2 about the stated line, and then use that matrix to find the reflection of the given point about that line.

The reflection of $(1, 2)$ about the line that makes an angle of $\pi/4$ $(= 45^\circ)$ with the positive x-axis.

Theorem 3.3.3 Theorem of Pythagoras in R^n

If $\mathbf{u}$ and $\mathbf{v}$ are orthogonal vectors in R^n with the Euclidean inner product, then $\|\mathbf{u} + \mathbf{v}\|^2 = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2$.